Parameter Estimation of Required Rate of Return on Stock

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Outline

Parameter estimation of Public Company

Dividend Discount Model

According to Williams (1938) for successive prices of a company, the following relation holds

$$P_t = (1 + k_t^\circ)P_{t-1} - d_t + u_t, \quad t = 1, 2, \dots, \tag{1}$$

- P_t is a stock price
- d_t is a dividend payment
- k_t° is a required rate of return on stock
- $u_t \sim \mathcal{N}(0, \sigma^2)$ is a sequence of i.i.d random errors

Parameter estimation of Public Company

DDM with covariates

As the required rate of return at time t may depends on macroeconomic variables and firm specific variables, such as, GDP, inflation, key financial ratios of the firm, and so on, we modeled the required rate of return by the following linear equation

$$k_t^\circ = k_1 + k_2 c_{2t} + \cdots + k_n c_{nt} = c_t' k, \quad t = 1, \dots, T, \tag{2}$$

- $k := (k_1, \dots, k_n)'$ is an $(n \times 1)$ parameter vector of the required rate of return
- $z_t := (c_{1t}, c_{2t}, \dots, c_{nt})'$ is an $(n \times 1)$ covariate vector at time t

Parameter estimation of Public Company

Parameter Estimators

Maximum likelihood estimators of the model are obtained by

$$\hat{k} := (X'X)^{-1}X'(p + d - p_{-1}) \quad \text{and} \quad \hat{\sigma}^2 := \frac{1}{T}e'e, \tag{3}$$

- $p := (p_1, \dots, p_T)'$ is a $(T \times 1)$ price vector,
- $d := (d_1, \dots, d_T)'$ is a $(T \times 1)$ dividend vector,
- $p_{-1} := (p_0, \dots, p_{T-1})'$ is a $(T \times 1)$ lagged price vector,
- $X' := [x_1 : \cdots : x_T]$ is an $(n \times T)$ matrix with $x_t := c_t P_{t-1}$.
- $e := p + d - p_{-1} - X\hat{k}$ is a $(T \times 1)$ unrestricted residual vector

Parameter estimation of Public Company

Main Result

If we assume $k_t^\circ > 0$, then the price process P_t is explosive time series. But we prove that

$$\frac{1}{\hat{\sigma}}(X'X)^{\frac{1}{2}}(\hat{k} - k) \xrightarrow{d} \mathcal{N}(0, I_n), \tag{4}$$

Main Result

Therefore, we have

$$\hat{k} \approx \mathcal{N}\big(k, \sigma^2(X'X)^{-1}\big). \tag{5}$$

As a result, all the theories of linear regression are asymptotically hold.

Parameter estimation of Public Company

DDM with Regime-Switching

DDM with N regimes is given by

$$P_t = \big(1 + k_t^\circ(s_t)\big)P_{t-1} - d_t + u_t = \big(1 + c_t'k(s_t)\big)P_{t-1} - d_t + u_t, \quad (6)$$

- s_t is an unobserved regime at time t, which is governed by a Markov chain with N states,
- $k(s_t)$ is an $(n \times 1)$ parameter vector of the required rate of return corresponding to the regime s_t

Parameter estimation of Public Company

DDM with Regime-Switching

First order means that

$$p_{ij} := \mathbb{P}(s_t = j | s_{t-1} = i) = \mathbb{P}(s_t = j | s_{t-1} = i, s_{t-2} = r_{t-2}, \dots, s_1 = r_1), \quad (7)$$

DDM with Regime-Switching

Transition probability matrix of the regime–switching process s_t is

$$\mathsf{P} = \begin{bmatrix} p_{11} & p_{12} & \dots & p_{1N} \\ p_{21} & p_{22} & \dots & p_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ p_{N1} & p_{N2} & \dots & p_{NN} \end{bmatrix}. \quad (8)$$

Parameter estimation of Public Company

Probabilistic Inference and Forecasting

$$z_{t|t} = \frac{(z_{t|t-1} \odot \eta_t)}{i_N'(z_{t|t-1} \odot \eta_t)} \quad \text{and} \quad z_{t+1|t} = \mathsf{P}' z_{t|t}, \quad t = 1, \dots, T, \tag{9}$$

- $z_{t|t} := \big(\mathbb{P}(s_t = 1|\mathcal{F}_t; \theta), \dots, \mathbb{P}(s_t = N|\mathcal{F}_t; \theta)\big)'$ is a probabilistic inference vector
- $z_{t+1|t} := \big(\mathbb{P}(s_{t+1} = 1|\mathcal{F}_t; \theta), \dots, \mathbb{P}(s_{t+1} = N|\mathcal{F}_t; \theta)\big)'$ is a forecast vector
- $\eta_t := (\eta_{t1}, \dots, \eta_{tN})'$ is a conditional density vector with

$$\eta_{tj} := \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{ -\frac{\big(P_t + d_t - \big(1 + c_t' k(j)\big) P_{t-1}\big)^2}{2\sigma^2} \right\} \tag{10}$$

Parameter estimation of Public Company

Smoothing

To obtain MLE of the population parameters, in addition to the inferences and forecasts we need smoothed inference about the regime–switching process was in at time t based on full information $\mathcal{F}_T$. The smoothed inferences can be obtained by using the Kim's (1994) smoothing algorithm:

$$z_{t|T} = z_{t|t} \odot \big\{\mathsf{P}'(z_{t+1|T} \oslash z_{t+1|t})\big\}, \quad t = 1, \dots, T, \tag{11}$$

- $z_{t|T} := \big(\mathbb{P}(s_t = 1|\mathcal{F}_T;\theta), \dots, \mathbb{P}(s_t = N|\mathcal{F}_T;\theta)\big)'$ is a smoothing inference vector
- $\oslash$ is an element–wise division of two vectors.

Parameter estimation of Public Company

Parameter Estimators

$$\hat{p}_{ij} = \frac{\sum_{t=2}^{T} \mathbb{P}\big(s_{t-1} = i, s_t = j | \mathcal{F}_T; \hat{\theta}\big)}{\sum_{t=2}^{T} (z_{t-1|T})_i}, \quad (12)$$

$$0 = \sum_{t=1}^{T} \left(\frac{\partial \ln(\eta_t)}{\partial \alpha'}\right)' z_{t|T}, \quad (13)$$

$$\hat{\rho} = z_{1|T}, \quad (14)$$

Parameter Estimators

For our model, we have

$$\hat{k}(j) = \big(\bar{X}_j' \bar{X}_j\big)^{-1} \bar{X}_j' \big(\bar{p}_j + \bar{d}_j - \bar{p}_{-1,j}\big), \quad \hat{\sigma}^2 = \frac{1}{T} \sum_{j=1}^{N} \bar{e}_j' \bar{e}_j \quad (15)$$

Parameter Estimation of Private Company

Main Equation

According DDM equation, $P_t = (1 + k_t^\circ)P_{t-1} - d_t$, one get that

$$(1 + b_t)mB_{t-1} = \big((1 + k_t^\circ)m - \Delta_t\big)B_{t-1}. \qquad (16)$$

- B_t is a book value of equity
- b_t is a book value growth rate
- Δ_t is a dividend–to–book ratio
- $m = P_t/B_t$ is a constant price–to–book ratio.

Parameter Estimation of Private Company

Linear Regression

As a result,

$$\Delta_t = mk_t^\circ - mb_t, \quad k_t^\circ = \frac{1}{m}\Delta_t + b_t \tag{17}$$

If we add random component to the equation, we obtain

$$b_t = c_t'k - \delta\Delta_t + u_t, \quad t = 1, \dots, T, \tag{18}$$

where $\delta := 1/m$ is a book–to–price ratio. Because the equation is a linear regression, all the regression theories hold.

Parameter Estimation of Private Company

State Space Representation

Now we assume that the price–to–book ratios are varies over time, that is, $m_t = P_t/B_t$, $t = 1, \dots, T$. Under the assumption, equation (16) becomes

$$m_t B_t = \big((1 + k_t^\circ) m_{t-1} - \Delta_t\big) B_{t-1}. \tag{19}$$

We assume that the price–to–book ratio is governed by AR(1) process. Then,

$$\begin{cases} \Delta_t = -(1 + b_t) m_t + (1 + k_t^\circ) m_{t-1} + u_t \\ m_t = \phi_0 + \phi_1 m_{t-1} + v_t \end{cases} \quad \text{for } t = 1, \dots, T. \tag{20}$$

Parameter Estimation of Private Company

State Space Representation

The systems is more compactly written by

$$\begin{cases} y_t = \psi_t' z_t + u_t \\ z_t = A z_{t-1} + a + \eta_t \end{cases} \qquad \text{for } t = 1, \dots, T, \tag{21}$$

- $z_t := (m_t, m_{t-1})'$ is a (2×1) state vector
- $\psi_t := \big(-(1+b_t), 1+k_t^\circ\big)'$ is a (2×1) vector
- $a := (\phi_0, 0)'$ is a (2×1) vector
- $\eta_t := (v_t, 0)'$ is a (2×1) random vector, and
- $$A := \begin{bmatrix} \phi_1 & 0 \\ 1 & 0 \end{bmatrix}$$

Kalman Filtering

Expectation (E) step

$$
\begin{aligned}
\Lambda(\theta|\mathcal{F}_T) &= \mathbb{E}\Big(\ln(f_{\bar{\Delta},\bar{m}}(\bar{\Delta},\bar{m}))\Big|\mathcal{F}_T\Big) \\
&= -\frac{2T+1}{2}\ln(2\pi) - \frac{T}{2}\ln(\sigma_u^2) \\
&-\frac{1}{2\sigma_u^2}\sum_{t=1}^{T}\mathbb{E}\bigg(\Big(\Delta_t + (1+b_t)m_t - (1+k_t^\circ)m_{t-1}\Big)^2\bigg|\mathcal{F}_T\bigg) \qquad (22) \\
&-\frac{T}{2}\ln(\sigma_v^2) - \frac{1}{2\sigma_v^2}\sum_{t=1}^{T}\mathbb{E}\bigg(\Big(m_t - \phi_0 - \phi_1 m_{t-1}\Big)^2\bigg|\mathcal{F}_T\bigg) \\
&-\frac{1}{2}\ln(\sigma_0^2) - \frac{1}{2\sigma_0^2}\mathbb{E}\bigg(\Big(m_0 - \mu_0\Big)^2\bigg|\mathcal{F}_T\bigg),
\end{aligned}
$$

Kalman Filtering

Maximization (M) step

$$\hat{k} := \left(\sum_{t=1}^{T}\left[e_1'\Gamma(z_{t-1}|T)e_1\right]c_tc_t'\right)^{-1} \times$$

$$\sum_{t=1}^{T} z_t\Big(\Delta_t e_1' z_{t-1|T} + (1+b_t)e_1'\Gamma(z_{t-1}, z_t|T)e_1 - e_1'\Gamma(z_{t-1}|T)e_1\Big),$$

$$\hat{\phi}_0 := \frac{\left[\sum_{t=1}^{T} e_1' z_{t|T}\right]\left[\sum_{t=1}^{T} e_1'\Gamma(z_{t-1}|T)e_1\right]}{T\left[\sum_{t=1}^{T} e_1'\Gamma(z_{t-1}|T)e_1\right] - \left[\sum_{t=1}^{T} e_1' z_{t-1|T}\right]^2},$$

$$\frac{-\left[\sum_{t=1}^{T} e_1' z_{t-1|T}\right]\left[\sum_{t=1}^{T} e_1'\Gamma(z_{t-1}, z_t|T)e_1\right]}{}$$

Kalman Filtering

Maximization (M) step

$$\hat{\phi}_1 := \frac{T\Big[\sum_{t=1}^T e_1'\Gamma(z_{t-1}, z_t|T)e_1\Big] - \Big[\sum_{t=1}^T e_1'z_{t-1|T}\Big]\Big[\sum_{t=1}^T e_1'z_{t|T}\Big]}{T\Big[\sum_{t=1}^T e_1'\Gamma(z_{t-1}|T)e_1\Big] - \Big[\sum_{t=1}^T e_1'z_{t-1|T}\Big]^2},$$

$$\hat{\mu}_0 := e_1'z_{0|0}, \quad \hat{\sigma}_0^2 := e_1'\Sigma(z_0|0)e_1$$

$$\hat{\sigma}_u^2 := \frac{1}{T}\sum_{t=1}^T \mathbb{E}\big(u_t^2|\mathcal{F}_T\big), \quad \hat{\sigma}_v^2 := \frac{1}{T}\sum_{t=1}^T \mathbb{E}\big(v_t^2|\mathcal{F}_T\big),$$

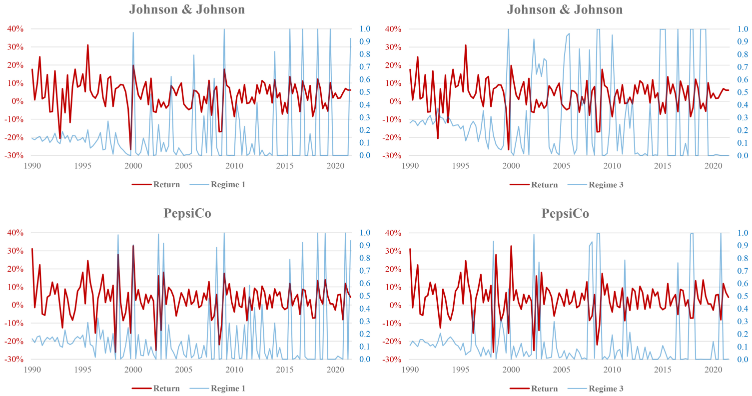
Numerical Results for Selected Firms

Table : Estimations of Parameters for the DDM

Row	Prmtrs	Johnson & Johnson			PepsiCo			JPMorgan		
2.	$k(j)$	10.43%	4.72%	–4.86%	11.39%	2.89%	–8.25%	15.37%	5.21%	–7.66%
3.	P	0.000	1.000	0.000	0.000	0.756	0.244	0.397	0.000	0.603
4.		0.000	0.762	0.238	0.077	0.812	0.111	0.080	0.323	0.598
5.		0.492	0.000	0.508	0.781	0.000	0.219	0.290	0.710	0.000
6.	τ_j	1.000	4.199	2.034	1.000	5.327	1.281	1.659	1.476	1.000
7.	π	0.138	0.581	0.281	0.169	0.681	0.150	0.232	0.393	0.375
8.	k_∞	2.82%			2.66%			2.74%		
9.	σ_3	3.007			3.238			5.029		
10.	k	2.55%			2.39%			2.88%		
11.	k_L	0.86%			0.28%			0.04%		
12.	k_U	4.24%			4.51%			5.72%		
13.	σ_1	7.262			8.400			9.219		

Numerical Results

Figure : Returns VS Regime Probabilities of Selected Companies



Conclusion

- In this paper, instead of the traditional CAPM and its descendant versions, we introduce new estimation methods: ordinary ML methods, Bayesian method, ML methods with regime–switching, and Kalman filtering to estimate the required rate of return.
- We estimate of price–to–book ratio m by the ordinary ML method and Bayesian method, price–to–book ratios with regime–switching $m(1), \dots, m(N)$ by ML method with regime–switching, and state (unobserved) variable of price–to–book ratio m_t by Kalman filtering method. For the Kalman filtering method, we develop EM algorithm.

Conclusion

- The parameter estimation methods of the franchise factor model can be used for not only private companies but also public companies.
- The ideas in the paper can be extended to multiple stocks, which are dependent using panel regression.

Thank you for your attention